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Hidden hierarchies of KdV type on Birkhoff strata ★

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Abstract

Integrable hierarchies arising from Schrödinger equations with energy-dependent potentials are found to determine flows on the strata of the Grassmannian different from the big cell. As a consequence they have wide classes of solutions associated with the zero sets of KdV τ-functions. The group-theoretical description of these hierarchies from the point of view of Birkhoff factorization theorem is given. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The relationship between the infinite-dimensional Grassmannian Gr [1–4] and the theory of integrable systems has been essential not only to explain the geometry of equations of Korteweg–de Vries (KdV) type but also to formulate several relevant problems arising in quantum field theory and string theory [5–9]. The Grassmannian has a decomposition into strata with a principal stratum, called the *big cell,* which is a dense open set in Gr. Integrable systems such as those of the Kadomtsev–Petviashvilii (KP), KdV or Gel'fand–Dikii hierarchies are described by flows in Gr such that their associated solutions blow up as the flows leave the big cell. For example, in the case of $\mathrm{Gr}^{(2)}$, the part of Gr which is relevant for the KdV hierarchy, every element $W \in \mathrm{Gr}^{(2)}$ determines a flow $W(t)$ in $\mathrm{Gr}^{(2)}$ and a solution of the hierarchy of the form

$$u_W(\boldsymbol{t}) = -2\frac{\partial^2 \ln\tau_W}{\partial t_1^2}(\boldsymbol{t}), \quad \boldsymbol{t} := (t_1, t_3, t_5 \ldots),$$

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where $\tau_W(\boldsymbol{t})$ is the τ-function associated to W. This solution is defined only for those $\boldsymbol{t}$ such that $\tau_W(\boldsymbol{t}) \neq 0$ and this is just the condition for $W(\boldsymbol{t})$ to be in the big cell of Gr. Due to the fact that the big cell is a dense open set of Gr, the flows $W(\boldsymbol{t})$ stay in the big cell for almost all $\boldsymbol{t}$. Nevertheless, the remaining strata of Gr are also of interest in the theory of integrable systems. For instance, in a recent work by Adler and van Moerbeke [10,11] the strata different from the big cell have been found to be essential to study the blow up behaviours and to regularize the solutions of the KP hierarchy near the blow up. The purpose of the present paper is to show that these strata are also useful to describe *hidden hierarchies* of integrable systems. More precisely, we consider the Grassmannian $\mathrm{Gr}^{(2)}$ which admits a partition into a numerable collection of strata of the form

$$\mathrm{Gr}^{(2)} = \bigcup_{m\geq 0} \Sigma_m,$$

where only the stratum Σ_0 is in the big cell. We find that the remaining strata $\Sigma_m, m \geq 1$, not only describe the singularities of the solutions of the KdV hierarchy but also support the flows of the integrable hierarchies associated to Schrödinger equations with energy-dependent potentials

$$\frac{\partial^2\psi}{\partial x^2} = \left(\lambda^{2m+1} + \sum_{n=0}^{2m} \lambda^n u_n(x)\right)\psi, \quad \lambda := k^2.$$

These hierarchies were introduced and studied from the point of view of the hamiltonian formalism in [13]. They were further generalised and analysed in [14–17]. In what follows they will be referred to as the *hidden* $(2m+1)$*th KdV hierachies* (hKdV_{2m+1}), since their flows take place outside the big cell.

The main ingredient of our analysis is the close link it establishes between the hKdV hierarchies and the zero sets of KdV τ-functions in the infinite-dimensional space $\mathbb{C}^\infty = \{\boldsymbol{t} = (t_1, t_3, t_5, \ldots), t_i \in \mathbb{C}\}$. Thus, it is proved that, as a function of t_1, $\tau_W(\boldsymbol{t})$ can have zeros of orders $l_m := m(m+1)/2$ $(m \geq 1)$ only, and that the set of l_m-order zeros of τ_W is characterized by some associated solutions of the $\mathrm{hKdV}_{(2m+1)}$ hierarchy. As a consequence, a method is provided for characterizing solutions of the hKdV hierarchies from τ-functions of the standard KdV hierarchy.

The stratification of the Grassmannian $\mathrm{Gr}^{(2)}$ has a group-theoretical formulation which derives from its representation as a homogeneous space

$$\mathrm{Gr}^{(2)} \cong L\mathrm{GL}_2/L^+\mathrm{GL}_2,$$

where $L\mathrm{GL}_2$ is the loop group of smooth maps from the unit circle to GL_2 and $L^+\mathrm{GL}_2$ is the subgroup of maps that extend holomorphically to the unit disk. From the Birkhoff factorization theorem [18] it follows that for any loop $g \in L\mathrm{GL}_2$ there exists a unique $\boldsymbol{m} = (m_1, m_2) \in \mathbb{Z}^2$, up to permutation, such that

$$g = g_- \cdot k^{\boldsymbol{m}} \cdot g_+$$

where

$$k^{\boldsymbol{m}} := \mathrm{diag}(k^{m_1}, k^{m_2})$$

and $g_{\pm} \in L^{\pm}\mathrm{GL}_2$, with $L^{-}\mathrm{GL}_2$ being the subgroup of maps that extend holomorphically outside the unit disk. In this picture the strata of $\mathrm{Gr}^{(2)}$ are identified with subgroups of $L^{-}\mathrm{GL}_2$ and the hKdV hierarchies turn out to be associated with the zero-curvature equations arising from Birkhoff factorization with $\boldsymbol{m} \neq \boldsymbol{0}$.

The paper is organized as follows. In Section 2 we introduce the hKdV hierarchies from the associated linear system. The peculiar form of the wave functions is revealed from the analysis of the asymptotic solutions of the corresponding equations of Schödinger type. Section 3 is devoted to the analysis of the zero sets of KdV τ- functions and their connection with the stratification of $\mathrm{Gr}^{(2)}$. In Section 4 it is shown how the hKdV hierarchies arise in $\mathrm{Gr}^{(2)}$ and a solution method for them is provided and illustrated through the consideration of some relevant examples as well as some applications to Calogero–Moser systems. Finally, Section 5 brings into play the loop group description of $\mathrm{Gr}^{(2)}$ in order to characterize the hKdV hierarchies in the context of Birkhoff factorization.

2. Integrable hierarchies associated to Shrödinger equations with energy-dependent potentials

Let us start by analysing the asymptotic solutions for large k of the Schrödinger spectral problem

$$\partial_x^2 \psi(k,x) = \left(\lambda^{2m+1} + \sum_{n=0}^{2m} \lambda^n u_n(x)\right) \psi(k,x), \quad \lambda := k^2, \tag{1}$$

where we use the notation $\partial_x f = \partial f/\partial x$. As usual we introduce the change of dependent variable

$$y = \partial_x \ln \psi,$$

which reduces (1) to the Ricatti equation

$$\partial_x y + y^2 = k^{4m+2} + \sum_{n=0}^{2m} k^{2n} u_n(x).$$

If we now assume a Laurent expansion for y

$$y = k^{2m+1}\left(1 + \sum_{n\geq 1} \frac{y_n(x)}{k^n}\right),$$

then by identifying the coefficients of the monomials k^n for $2m \leq n \leq 4m+1$ in the Ricatti equation the asymptotic solution of (1) can be written as

$$\psi(k,x) = \exp\left[\sum_{n=1}^{m} k^{2n-1} b_n(x) + k^{2m+1} x\right]\left(1 + \sum_{n\geq 1} \frac{a_n(x)}{k^n}\right).$$

It must be noticed the presence of the m functions $b_n(x)$ which characterize the factor of ψ with an essential singularity at $k = \infty$. This feature establishes an important difference with respect to the standard Schrödinger spectral problem ($m = 0$).

Our next task is to describe the $\mathrm{hKdV}_{(2m+1)}$ hierarchy associated with (1). To this end it is convenient to introduce the following infinite set of variables

$$\boldsymbol{t}_m := (t_{2m+1}, t_{2m+3}, t_{2m+5}, \ldots),$$

and in particular we will henceforth denote

$$x := t_{2m+1}.$$

We are going to characterize the members of the $\mathrm{hKdV}_{(2m+1)}$ hierarchy in terms of solutions of an infinite linear system of the form

$$\partial_x^2 \psi = u(\lambda, \boldsymbol{t}_m)\psi, \tag{2}$$

$$\partial_{2n+1}\psi = \alpha_n(\lambda, \boldsymbol{t}_m)\psi + \beta_n(\lambda, \boldsymbol{t}_m)\partial_x\psi, \quad n > m, \qquad \partial_{2n+1} := \frac{\partial}{\partial t_{2n+1}}, \tag{3}$$

where $u = u(\lambda, \boldsymbol{t}_m)$ is defined by

$$u := \lambda^{2m+1} + \sum_{n=0}^{2m} \lambda^n u_n(\boldsymbol{t}_m),$$

and α_n and β_n are polynomials in λ. Moreover, as $k \to \infty$ the wave function ψ is assumed to admit an expansion

$$\psi(k, \boldsymbol{t}_m) = \exp\left[\sum_{n=1}^{m} k^{2n-1} b_n(\boldsymbol{t}_m) + \sum_{n \geq m+1} k^{2n-1} t_{2n-1}\right]\left(1 + \sum_{n \geq 1} \frac{a_n(\boldsymbol{t}_m)}{k^n}\right). \tag{4}$$

First of all we introduce the following bilinear form for functions of k:

$$B(\phi, \psi) := -\frac{\phi(k)\psi(-k) - \phi(-k)\psi(k)}{2k^{2m+1}}.$$

Then, Eq. (3) for $\psi(k, \boldsymbol{t}_m)$ and $\psi(-k, \boldsymbol{t}_m)$ leads to a linear system for α_n and β_n, the solution of which is:

$$\alpha_n = \frac{B(\partial_{2n+1}\psi, \partial_x\psi)}{B(\psi, \partial_x\psi)}, \qquad \beta_n = \frac{B(\psi, \partial_{2n+1}\psi)}{B(\psi, \partial_x\psi)}. \tag{5}$$

Moreover, from (2) it immediately follows

$$\partial_x B(\psi, \partial_x\psi) \equiv 0,$$

and hence it is easy to see that

$$\partial_x \beta_n = -2\alpha_n + \partial_{2n+1} \ln B(\psi, \partial_x\psi).$$

If we now take into account that

$$B(\psi, \partial_x\psi) = 1 + \mathrm{O}(\lambda^{-1}), \tag{6}$$

then as α_n and β_n are polynomials in λ, from (6) we deduce

$$\partial_{2n+1} B(\psi, \partial_x \psi) \equiv 0,$$

and

$$\alpha_n = -\tfrac{1}{2}\partial_x \beta_n. \tag{7}$$

Furthermore, the compatibility condition for (2) and (3) implies

$$\begin{aligned}\partial_{2n+1}(\partial_x^2\psi - u\psi) = (&-\partial_{2n+1}u + \partial_x^2\alpha_n + 2u\partial_x\beta_n + \beta_n\partial_x u)\psi \\ &+(2\partial_x\alpha_n + \partial_x^2\beta_n)\partial_x\psi = 0.\end{aligned}$$

Hence, we obtain

$$\partial_{2n+1}u = -\tfrac{1}{2}\partial_x^3\beta_n + 2u\partial_x\beta_n + \beta_n\partial_x u. \tag{8}$$

The final step consists in relating the function β_n with the trace of the resolvent of the Schrödinger operator

$$R(\lambda, \boldsymbol{t}_m) := \frac{\psi(k, \boldsymbol{t}_m)\psi(-k, \boldsymbol{t}_m)}{B(\psi, \partial_x\psi)}.$$

Indeed, by taking into account that

$$-2k^{2m+1}B(\psi, \partial_{2n+1}\psi) = -2k^{2n+1}\psi(k)\psi(-k) + \mathrm{O}(k^{-1}),$$

then from (5) and the polynomial character of β_n as a function of λ it follows that

$$\beta_n = (\lambda^{n-m}R)_+.$$

Here $(\lambda^{n-m}R)_+$ stands for the polynomial part of $\lambda^{n-m}R$ with R being substituted by its expansion as $\lambda \to \infty$

$$R = 1 + \sum_{n\geq 0}\frac{R_n(\boldsymbol{t})}{\lambda^n}.$$

It turns out [13] that the coefficients R_n are differential polynomials in the potential coefficients $(u_0, u_1, \ldots, u_{2m})$ which can be determined by means of the equation

$$-\tfrac{1}{2}\partial_x^3 R + 2u\partial_x R + (\partial_x u)R = 0.$$

Therefore, Eq. (8) can be expressed as

$$\partial_{2n+1}u = (-\tfrac{1}{2}\partial_x^3 + 2u\partial_x + \partial_x u)(\lambda^{n-m}R)_+. \tag{9}$$

Now, due to the fact that

$$(-\tfrac{1}{2}\partial_x^3 + 2u\partial_x + \partial_x u)(\lambda^{n-m}R)_+ = -(-\tfrac{1}{2}\partial_x^3 + 2u\partial_x + \partial_x u)(\lambda^{n-m}R)_-,$$

it is immediate to realize that Eq. (9) constitutes an evolution equation for $(u_0, u_1, \ldots, u_{2m})$. The set of these equations is the $\mathrm{hKdV}_{(2m+1)}$ hierarchy of integrable equations associated with the Schrödinger operator (1). Solutions of the members of the hierarchies can be derived from the functions b_i and a_n arising in expansion (4).

For example, the potential function for the $\mathrm{hKdV}_{(3)}$ hierarchy is given by

$$\begin{aligned} u_0 &= 2\partial_{x a_3} - 2a_2\partial_x a_1 + a_1\partial_x^2 b_1 + 2\partial_x a_1 \partial_x b_1, \\ u_1 &= (\partial_x b_1)^2 + 2\partial_x a_1, \qquad u_2 = 2\partial_x b_1. \end{aligned} \tag{10}$$

The analogous formulas for the $\mathrm{hKdV}_{(5)}$ hierarchy are

$$\begin{aligned} u_0 &= 2\partial_x a_5 - a_4\partial_x a_1 + a_1\partial_x^2 b_1 + a_3\partial_x^2 b_2 \\ &\quad + 2\partial_x b_1\partial_x a_1 + 2a_3\partial_x b_2 - a_2(\partial_x b_1)^2 - 2a_2\partial_x a_3 \\ &\quad + 2a_2^2\partial_x a_1 - a_1 a_2\partial_x^2 b_2 - 2a_2\partial_x b_2\partial_x a_1, \\ u_1 &= (\partial_x b_1)^2 + 2\partial_x a_3 - 2a_2\partial_x a_1 + a_1\partial_x^2 b_2 + 2\partial_x a_1\partial_x b_2, \\ u_2 &= 2\partial_x b_1\partial_x b_2 + 2\partial_x a_1, \quad u_3 = 2\partial_x b_1 + (\partial_x b_2)^2, \quad u_4 = 2\partial_x b_2. \end{aligned}$$

The first equation of the $\mathrm{hKdV}_{(3)}$ hierarchy corresponds to the time parameter $t = t_5$ and takes the form

$$\begin{aligned} \partial_t u_0 &= \tfrac{1}{4}\partial_x^3 u_2 - u_0\partial_x u_2 - \tfrac{1}{2}u_2\partial_x u_0, \\ \partial_t u_1 &= -\tfrac{1}{2}u_2\partial_x u_1 - u_1\partial_x u_2 + \partial_x u_0, \\ \partial_t u_2 &= -\tfrac{3}{2}u_2\partial_x u_2 + \partial_x u_1. \end{aligned} \tag{11}$$

The simplest member of the $\mathrm{hKdV}_{(5)}$ hierarchy is associated with the time parameter $t = t_7$ and can be written as

$$\begin{aligned} \partial_t u_0 &= \tfrac{1}{4}\partial_x^3 u_4 - u_0\partial_x u_4 - \tfrac{1}{2}u_4\partial_x u_0, \\ \partial_t u_1 &= -\tfrac{1}{2}u_4\partial_x u_1 - u_1\partial_x u_4 + \partial_x u_0, \\ \partial_t u_2 &= -\tfrac{1}{2}u_4\partial_x u_2 - u_2\partial_x u_4 + \partial_x u_1, \\ \partial_t u_3 &= -\tfrac{1}{2}u_4\partial_x u_3 - u_3\partial_x u_4 + \partial_x u_2, \\ \partial_t u_4 &= -\tfrac{3}{2}u_4\partial_x u_4 + \partial_x u_3. \end{aligned}$$

3. Zeros of τ-functions and the stratification of the Grassmannian

Let $H = L^2(S^1)$ be the Hilbert space of all square-integrable functions on the unit circle S^1 of the complex plane. It can be decomposed as the direct sum $H = H_+ \oplus H_-$ of the closed subspaces

$$H_+ := \overline{\mathbb{C}\{k^n\}}_{n\geq 0}, \qquad H_- := \overline{\mathbb{C}\{k^{-n}\}}_{n\geq 1}.$$

We will consider the Grassmannian Gr of all closed subspaces W of H such that

(i) The orthogonal projections $P_\pm : W \to H_\pm$ are operators of Fredholm and compact types, respectively.

(ii) The virtual dimension of W (i.e. the index of P_+) is zero.

It can be proved that Gr constitutes a connected Banach manifold – if P_- is Hilbert–Schmidt then Gr is a Hilbert manifold – which exhibits a stratified structure. To describe the strata of Gr it is required to introduce the set S_0 of increasing sequences of integers

$$S = \{s_0, s_1, s_2, \dots\},$$

such that $s_n = n$ for all sufficiently large n. Each $W \in$Gr determines a sequence of this type. To see this point recall that an element $w \in H$ is said to be of finite order n if it can be expressed in the form $w = \Sigma_{m\leq n} a_m k^m$, with $a_n \neq 0$. Thus, due to the fact that the virtual dimension of W is zero, it can be shown that the sequence

$$S_W = \{n \in \mathbb{Z}\text{: } W \text{ contains an element of order } n\}$$

is an element of S_0. Thus, given $S \in S_0$ we may define the subset of Gr

$$\Sigma_S = \{W \in \mathrm{Gr}\text{: } S_W = S\},$$

which is called the stratum corresponding to S. In any $W \in$Gr the elements of finite order form a dense open subspace denoted by W^{alg}. Therefore, W belongs to Σ_S when W^{alg} has a basis $\{w_n\}_{\geq 0}$ such that

$$w_n(k) = k^{s_n}(1 + \mathrm{O}(k^{-1})), \quad n \geq 0.$$

The stratum Σ_S is a submanifold of Gr of finite codimension given by

$$\mathrm{codim}\Sigma_S = \sum_{n\geq 0}(n - s_n).$$

In particular, if S is the set of non-negative integers, the corresponding stratum has codimension zero and constitutes a dense open subset of Gr which is called *the big cell* of the Grassmannian.

In the analysis of the KdV hierarchy one is lead to consider the subset of Gr given by

$$\mathrm{Gr}^{(2)} = \{W \in \mathrm{Gr}\text{: } k^2 W \subset W\}.$$

Here k^2 denotes the action of the multiplication operator by the function K^2. It is obvious that $S_W + 2 \subset S_W$ for all $W \in \mathrm{Gr}^{(2)}$, and as a consequence the stratification of $\mathrm{Gr}^{(2)}$ turns out to be

$$\mathrm{Gr}^{(2)} = \bigcup_{m\geq 0} \Sigma_m, \qquad \Sigma_m := \Sigma_{S_m} \cap \mathrm{Gr}^{(2)}, \tag{12}$$

where

$$S_m = \{-m, -m+2, -m+4, \dots, m, m+1, m+2, \dots\}.$$

The KdV flow on $Gr^{(2)}$ is defined for appropriate $\boldsymbol{t} := (t_1, t_3, t_5, \dots) \in \mathbb{C}^\infty$ in terms of the trajectories

$$W(\boldsymbol{t}) := \psi_0(k, \boldsymbol{t})^{-1} W = \{\psi_0(k, \boldsymbol{t})^{-1} w : w \in W\}, \quad W \in \mathrm{Gr}^{(2)},$$

where

$$\psi_0(k, \boldsymbol{t}) := \exp\left(\sum_{n\geq 0} t_{2n+1}k^{2n+1}\right)$$

is the so called *vacuum wave function* of the KdV hierarchy.

As it is proved in [2] $W(\boldsymbol{t})$ belongs to the stratum Σ_0 for *almost all* $\boldsymbol{t}$. This is so because there exists a non-zero holomorphic function $\tau_W(\boldsymbol{t})$ associated with W such that the function defined by

$$\psi_W(k, \boldsymbol{t}) := \psi_0(k, \boldsymbol{t})\frac{\tau_W(\boldsymbol{t} - \epsilon(k))}{\tau_W(\boldsymbol{t})},$$

with

$$\epsilon(k) := \left(\frac{1}{k}, \frac{1}{3k^3}, \ldots, \frac{1}{(2n+1)k^{2n+1}}, \ldots\right),$$

belongs to W for all $\boldsymbol{t}$ such that $\tau_W(\boldsymbol{t}) \neq 0$. In this way, and taking into account that the derivatives of f with respect to the variables t_n are also members of W, it is trivial to prove that provided $\tau_W(\boldsymbol{t}) \neq 0$ the subspace $W(\boldsymbol{t})$ contains elements w_n of order n for all $n \geq 0$. Therefore, $W(\boldsymbol{t}) \in \Sigma_0$ and, as a consequence, there exist decompositions of the form

$$\begin{aligned} \partial_x^2\psi_W &= (\lambda + u_w(\boldsymbol{t}))\psi_W, \quad x := t_1, \quad \lambda := k^2, \\ \partial_{2n+1}\psi_W &= \alpha_n(k^2, \boldsymbol{t})\psi_W + \beta_n(k^2, \boldsymbol{t})\partial_x\psi_W, \end{aligned}$$

where

$$\partial_{2n+1} := \frac{\partial}{\partial t_{2n+1}}, \quad n \geq 0,$$

and α_n and β_n are polynomials in k^2. By imposing the compatibility between these equations one gets the standard KdV hierarchy of evolution equations for the function

$$u_W(\boldsymbol{t}) = -2\partial_1^2 \ln \tau_W(\boldsymbol{t}).$$

Now we consider one of the main points of our discussion: the analysis of the zero set of $\tau_W(\boldsymbol{t})$ or, equivalently, the set of singularities of the corresponding solution $u_W(\boldsymbol{t})$ of the KdV hierarchy.

Theorem 1. *Let* $\boldsymbol{t}_0 = (t_{01}, t_{03}, t_{05}, \ldots)$ *be a zero of* $\tau_W(\boldsymbol{t})$. *Then, there exists an integer* $m > 0$ *such that the function* $\tau_W(t_1, t_{03}, t_{05}, \ldots)$ *has a zero of order*

$$l_m := \frac{m(m-1)}{2},$$

at $t_1 = t_{01}$.

Proof. As $W \in \mathrm{Gr}^{(2)}$ it follows that $W(\boldsymbol{t}_0) \in \mathrm{Gr}^{(2)}$. Hence, from Eq. (12) and taking into account that $\tau_W(\boldsymbol{t}_0) = 0$, the subspace $W(\boldsymbol{t}_0)$ is in one of the strata Σ_m for some $m > 0$. Now, from Proposition 8.6 of [2] we have that for any $V \in \mathrm{Gr}$

$$\tau_V(t_1, 0, 0, \ldots) = ct_1^l + \mathrm{O}(t_1^{l+1}),$$

where $c \neq 0$ and l is the codimension of the stratum of Gr containing V. Moreover, it is easy to find that

$$\operatorname{codim} \Sigma_{S_m} = l_m,$$

and therefore

$$\tau_{W(t_0)}(x, 0, 0, \dots) = c x^{l_m} + \mathrm{O}(x^{l_m+1}),$$

with $c \neq 0$. Furthermore, according to the following property of τ-functions, which derives from Eq. (3.4) of [2],

$$\tau_V(\boldsymbol{t} + \boldsymbol{t}') = \tau_{V(\boldsymbol{t})}(\boldsymbol{t}'),$$

we have

$$\tau_W(t_{01} + x, t_{03}, t_{05}, \dots) = \tau_{W(t_0)}(x, 0, 0, \dots).$$

Hence, the statement of the theorem follows at once. □

As a consequence of this result we see that the minimal order l_m for which a derivative of the form $\partial_1^{l_n} \tau_W(\boldsymbol{t}_0)$ does not vanish characterizes the stratum Σ_m containing $W(\boldsymbol{t}_0)$. Thus, we may state the following result.

Corollary 1. *The following statements are equivalent:*

(1) $W(\boldsymbol{t}_0)$ *is in the stratum* Σ_m.

(2) *The* τ*-function of* W *satisfies*

$$\partial_1^n \tau_W(\boldsymbol{t}_0) = 0, \quad 0 \leq n < l_m, \quad \partial_1^{l_m} \tau_W(\boldsymbol{t}_0) \neq 0.$$

(3) *The* τ*-function of* W *satisfies*

$$\partial_1^{l_n} \tau_W(\boldsymbol{t}_0) = 0, \quad 0 \leq n < m, \quad \partial_1^{l_m} \tau_W(\boldsymbol{t}_0) \neq 0.$$

4. Zeros of τ-functions and hKdV hierarchies

We are now in position to analyse the relationship between the zero sets of τ-functions and the hKdV hierarchies. Let us suppose given $W \in \mathrm{Gr}^{(2)}$ and let us denote by Z_W the zero set of the corresponding τ-function τ_W. According to Theorem 1 there is a partition of Z_W of the form

$$Z_W = \bigcup_{m \geq 1} Z_W^{l_m},$$

where $Z_W^{l_m}$ stands for the set of zeros $\boldsymbol{t}_0 = (t_{01}, t_{03}, t_{05}, \dots)$ of $\tau_W(\boldsymbol{t})$ such that the function $\tau_W(t_1, t_{03}, t_{05}, \dots)$ has a zero of order l_m at $t_1 = t_{01}$. From Corollary 1 we see that $Z_W^{l_m}$ can be characterized as the set of solutions $\boldsymbol{t} \in \mathbb{C}^\infty$ of the system of m equations

$$\partial_1^{l_n} \tau_W(\boldsymbol{t}) = 0, \quad 0 \leq n < m, \tag{13}$$

satisfying $\partial_1^{l_m} \tau_W(\boldsymbol{t}) \neq 0$.

The set Z_W is an *analytic set* in $\mathbb{C}^\infty$ [19], so that it can be considered as a union of complex manifolds. Suppose we are able to find a *patch* in Z_W described by a mapping $\mathbb{D} \subset \mathbb{C}^\infty \to Z_W^{l_m}$ of the following form:

$$\boldsymbol{t}_m := (t_{2m+1}, t_{2m+3}, t_{2m+5}, \dots) \mapsto \boldsymbol{t}(\boldsymbol{t}_m) := (b_1(\boldsymbol{t}_m), \dots, b_m(\boldsymbol{t}_m), \boldsymbol{t}_m), \tag{14}$$

where the functions b_i are m complex-valued functions depending on $\boldsymbol{t}_m$. This means that $\boldsymbol{t}(\boldsymbol{t}_m)$ is required to satisfy Eq. (13) and $\partial_1^{l_m} \tau_W(\boldsymbol{t}(\boldsymbol{t})) \neq 0$ for all $\boldsymbol{t}_m \in \mathbb{D}$. Notice that the functions b_i can be found by solving (13) with respect to the first m variables t_{2i+1}.

We are going to see that patches $\boldsymbol{t}(\boldsymbol{t}_m)$ are associated with solutions of the hKdV$_{(2m+1)}$ hierarchy. From Corollary 1 we have that $W(\boldsymbol{t}(\boldsymbol{t}_m)) \in \Sigma_m$ for all $\boldsymbol{t}_m \in \mathbb{D}$, and therefore there exists a unique function in $W(\boldsymbol{t}(\boldsymbol{t}_m))$ of order-m

$$\hat{\psi}_W(k, \boldsymbol{t}_m) = \frac{1}{k^m}\left(1 + \frac{a_1(\boldsymbol{t}_m)}{k} + \cdots + \frac{a_n(\boldsymbol{t}_m)}{k^n} + \cdots\right). \tag{15}$$

Theorem 2. *The function*

$$\psi_W(k, \boldsymbol{t}_m) = \psi_0(k, \boldsymbol{t}(\boldsymbol{t}_m))\hat{\psi}_W(k, \boldsymbol{t}_m), \tag{16}$$

satisfies the linear system (2), (3).

Proof. From expansion (15) we have that for all $n \geq 0$ the functions

$$k^{2n}\psi_0^{-1}\psi_W, \qquad k^{2n}\psi_0^{-1}\partial_x\psi_W, \tag{17}$$

are elements of $W(\boldsymbol{t}(\boldsymbol{t}_m))$ of orders $2n - m$ and $2n + m + 1$, respectively. Hence, due to the fact that $W(\boldsymbol{t}(\boldsymbol{t}_m)) \in \Sigma_m$, it follows that the functions in (17) form a basis of $W(\boldsymbol{t}(\boldsymbol{t}_m))^{\mathrm{alg}}$.

Moreover, by denoting

$$b(\boldsymbol{t}_m) := \sum_{n=1}^{m} k^{2n-1} b_n(\boldsymbol{t}_m),$$

and

$$\eta := \partial_x^2\psi_W - (\partial_x b + k^{2m+1})^2\psi_W - (2k^{2m}\partial_x a_1)\psi_W,$$

it is obvious that $\eta \in W$. Furthermore

$$\psi_0^{-1}\eta = \mathrm{O}(k^{m-1}),$$

so that $\psi_0^{-1}\eta$ belongs to $W(\boldsymbol{t}(\boldsymbol{t}_m))$ and has an order not greater than $m - 1$. Hence, there exists a decomposition

$$\psi_0^{-1}\eta = \sum_{n=0}^{m-1} k^{2n} c_n(\boldsymbol{t}_m)\psi_0^{-1}\psi_W,$$

and this implies Eq. (2).

In a similar way one proves that (16) satisfies (3). □

The above analysis provides a method for generating solutions to the hKdV hierarchies from elements $W \in \mathrm{Gr}^{(2)}$. The starting point is the τ-function $\tau_W(\boldsymbol{t})$ corresponding to W. Suppose that for a given $m \geq 1$ the system (13) can be solved with respect to $(t_1, \ldots, t_{2m-1})$ in terms of m functions of $\boldsymbol{t}_m = (t_{2m+1}, t_{2m+3}, \ldots)$

$$t_{2i-1} = b_i(\boldsymbol{t}_m), \quad i = 1, \ldots, m.$$

Then, the function (16) determines a wave function of the $\mathrm{hKdV}_{(2m+1)}$ hierarchy on the domain $\mathbb{D}$ of points $\boldsymbol{t}_m$ such that

$$\partial_1^{l_m} \tau_W(\boldsymbol{t}(\boldsymbol{t}_m)) \neq 0.$$

Our next theorem shows how to determine the explicit form of (15) from the τ-function of W.

Theorem 3. *If $W(\boldsymbol{t}_0)$ is in the stratum Σ_m then the function $\hat{\psi}_W(k, \boldsymbol{t}_0)$ is given by*

$$\hat{\psi}_W(k, \boldsymbol{t}_0) = \frac{\partial_1^{l_{m-1}} \tau_W(\boldsymbol{t}_0 - \boldsymbol{\epsilon}(k))}{\partial^{l_{m-1}} P_m(\boldsymbol{\partial}) \tau_W(\boldsymbol{t}_0)}, \tag{18}$$

where $P_m(\boldsymbol{\partial})$, $\boldsymbol{\partial} := (\partial_1, \partial_3, \ldots)$, is obtained from the identity

$$\exp(-\boldsymbol{\epsilon}(k) \cdot \boldsymbol{\partial}) = \sum_{n \geq 0} \frac{1}{k^n} P_n(\boldsymbol{\partial}).$$

Proof. The proof of this result is based on the properties of the decomposition of τ-functions in terms of Schur functions [2]

$$\tau_V(\boldsymbol{t}) = \sum_S v^S F_S(\boldsymbol{t}). \tag{19}$$

Each F_S is a polynomial in the t_i, homogeneous of weight $l(S)$ if we give t_i weight i, with $l(S)$ given by the codimension of the stratum Σ_S. It turns out [2] that the minimal weight of the terms in (19) is the codimension of the stratum on which V lies. hence, by taking into account that

$$\tau_W\left(t_{01} - \frac{1}{k} + x, t_{03} - \frac{1}{3k^3}, t_{05} - \frac{1}{5k^5}, \ldots\right)$$
$$= \tau_{W(\boldsymbol{t}_0)}\left(x - \frac{1}{k}, -\frac{1}{3k^3}, -\frac{1}{5k^5}, \ldots\right),$$

from the assumption $W(\boldsymbol{t}_0) \in \Sigma_m$ we deduce that

$$\partial_1^n \tau_W(\boldsymbol{t}_0 - \boldsymbol{\epsilon}(k)) = \mathrm{O}\left(\frac{1}{k^{l_m - n}}\right). \tag{20}$$

Hence, the minimal order $n_{\min}$ for which $\partial_1^{n_{\min}} \tau_W(\boldsymbol{t}_0 - \boldsymbol{\epsilon}(k))$, as a function of k, is not identically zero must satisfy $n_{\min} \geq l_{m-1}$. Otherwise $W(\boldsymbol{t}_0)$ would admit elements of degree $d < m$ and this would contradict the assumption $W(\boldsymbol{t}_0) \in \Sigma_m$. Let us see that

$n_{\min} = l_{m-1}$. Firstly, we notice that $n_{\min}$ is of the form l_p for some $p \geq 0$. This follows from Theorem 1 which implies that $n_{\min}$ is the minimum of the values l_p corresponding to the strata Σ_p such that $W(\boldsymbol{t}_0 - \epsilon(k)) \in \Sigma_p$ for some value of k. Moreover, as $\partial_1^{l_m} \tau_W(\boldsymbol{t}_0) \neq 0$ and $P_2(\boldsymbol{\partial}) = \partial_1^2/2$ it follows easily that $\partial_1^{l_m-2} \tau_W(\boldsymbol{t}_0 - \varepsilon(k)) \equiv 0$, so that $n_{\min}$ is of the form l_p with $p < m$. Therefore $n_{\min} = l_{m-1}$.

Finally, from (20) we deduce

$$\partial_1^{l_m-1} \tau_W(\boldsymbol{t}_0 - \varepsilon(k)) = \frac{c}{k^m} + \mathrm{O}\left(\frac{1}{k^{m+1}}\right),$$

with $c \neq 0$, since otherwise $W(\boldsymbol{t}_0)$ would admit elements of degree $d < m$. The rest of the proof follows at once. □

In view of the above results we have that the known classes of τ-functions for the standard KdV hierarchy are to our disposal in order to generate solutions to the hKdV hierarchies. For example, we can take the class which characterizes the rational solutions (see [2–4] and [20–22] vanishing as $x \to \infty$. These τ-functions can be obtained by means of coordinate translations from the τ-functions τ_m associated with the subspaces

$$H_m = \overline{\mathbb{C}\{k^s\}_{s \in S_m}}.$$

They can be written in the form

$$\tau_m(\boldsymbol{t}) = \begin{vmatrix} h_m & h_{m+1} & \cdots & h_{2m-1} \\ h_{m-2} & h_{m-1} & \cdots & h_{2m-3} \\ \vdots & \vdots & & \vdots \\ h_{2-m} & h_{3-m} & \cdots & h_1 \end{vmatrix}, \quad m \geq 1,$$

where $h_i = h_i(\boldsymbol{t})$ are the Schur polynomials:

$$\exp\left(-\sum_{n \geq 0} t_{2n+1} k^{2n+1}\right) = \sum_{i \geq 0} h_i(\boldsymbol{t}) k^i,$$

and $h_i \equiv 0$ for $i < 0$. The first few are

$$\tau_1 = -t_1, \qquad \tau_2 = -\tfrac{1}{3} t_1^3 + t_3, \qquad \tau_3 = \tfrac{1}{45} t_1^6 - \tfrac{1}{3} t_1^3 t_3 + t_1 t_5 - t_3^2.$$

Let us describe some solutions of the $\mathrm{hKdV}_{(2m+1)}$ hierarchies for $m = 1, 2$ which derive from these τ-functions.

We first consider τ_2 which can be factorized as

$$\tau = -\frac{1}{3} \prod_{i=0}^{2} (t_1 - \varepsilon^i \sqrt[3]{3t_3}), \quad \varepsilon := \exp\left(\mathrm{i}\frac{2\pi}{3}\right).$$

Thus, for $t_3 \neq 0$ we have three patches $\boldsymbol{t}^{(i)}(t_1)$ $(i = 0, 1, 2)$ with associated functions

$$b_1^{(i)} = \epsilon^i \sqrt[3]{3t_3}.$$

Each of them determines a wave function of the $\mathrm{hKdV}_{(3)}$ hierarchy. For example, for $i=0$ we get

$$\psi_W = \exp\left(k\sqrt[3]{3x} + k^3 x + \sum_{n\geq 2} t_{2n+1}k^{2n+1}\right)\left(\frac{1}{k} - \frac{1}{k^2}\frac{1}{\sqrt[3]{3x}}\right), \quad x := t_3,$$

and the following solution of the $\mathrm{hKdV}_{(3)}$ hierarchy:

$$u_0 = \frac{4}{x^2}, \qquad u_1 = \frac{3}{(3x)^{4/3}}, \qquad u_2 = \frac{2}{(3x)^{2/3}}.$$

The analysis of the solutions of the hKdV hierarchies provided by τ_3 is more involved. The discriminant of τ_3 with respect to t_1 is

$$\Delta(\boldsymbol{t}_1) = [(3t_3)^5 - (5t_5)^3]^2.$$

Hence, if $\Delta(\boldsymbol{t})_1 \neq 0$ the polynomial $\tau_3(t_1, \boldsymbol{t}_1)$, as a function of t_1, has simple roots only, so that we may define six patches $\boldsymbol{t}^{(i)}(\boldsymbol{t}_1)$ which lead to solutions of the $\mathrm{hKdV}_{(3)}$ hierarchy. The corresponding functions $t_1 = b_1^{(i)}(\boldsymbol{t}_1)$ satisfy the constraint

$$t_3^2 + \tfrac{1}{3}t_1^3 t_3 - \tfrac{1}{45}t_1^6 - t_1 t_5 = 0,$$

which can be explicitly solved for the variable $x := t_3$ as

$$x = -\frac{t_1^3}{6} \pm \sqrt{\frac{t_1^6}{20} + t_1 t_5}.$$

Thus, one finds two real continuous branches $t_1 = b_1^{(a)}(x, t_5)(a = 1, 2)$. Observe that the branch over the point $(x, t_1) = ((5t_5)^{3/5}/3, (5t_5)^{1/5})$, has singular x-derivative at that point. Notice also that $b^{(2)}(c, t_5) = -b^{(1)}(-x, -t_5)$.

Let us consider now the case $\Delta(\boldsymbol{t}_1) = 0$; that is to say,

$$t_3 = \tfrac{1}{3}(5t_5)^{3/5},$$

for a certain determination of the quintic root. Under this condition one finds

$$\tau_3 = \frac{1}{45}(t_1 - (5t_5)^{1/5})^3 \prod_{i=0}^{2}(t_1 - a_i(5t_5)^{1/5}),$$

where a_i stand for the three different roots of $a^3 + 3a^2 + 6a + 5$. Thus we get a patch $\boldsymbol{t}(\boldsymbol{t}_2)$ determined by

$$b_1 = (5x)^{1/5}, \quad b_2 = \tfrac{1}{3}(5x)^{3/5}, \qquad x := t_5.$$

Observe that $\tau_3(\boldsymbol{t}(\boldsymbol{t}_2) - \epsilon(k)) \equiv 0$, and that the corresponding wave function is

$$\psi_W(k, \boldsymbol{t}_2) = \exp(k(5x)^{1/5} + k^3\tfrac{1}{3}(5x)^{3/5} + k^5 x + \cdots) \times\left(\frac{1}{k^2} - \frac{(5x)^{-1/5}}{k^3}\right).$$

The associated solution of the $\mathrm{hKdV}_{(5)}$ hierarchy is

$$u_0 = \frac{6}{5x^2}, \qquad u_1 = \frac{5}{(5x)^{8/5}}, \qquad u_2 = \frac{4}{(5x)^{6/5}},$$
$$u_3 = \frac{3}{(5x)^{4/5}}, \qquad u_4 = \frac{2}{(5x)^{2/5}}.$$

The τ-functions of the KdV hierarchy of polynomial type are relevant in the analysis of the motion of poles for the rational solutions of the KdV equation [22]

$$\partial_3 u = \partial_1^3 u - 6u\partial_1 u.$$

Suppose $\tau_W(t_1, \boldsymbol{t}_1)$ is one of these functions. From the results of [2,22] one may prove that for most values of $\boldsymbol{t}_1$ there exists a positive integer m such that τ_W can be factorized into l_m different simple factors as

$$\tau_W(t_1, \boldsymbol{t}_1) = \prod_{i=1}^{lm} (t_1 - p_i(\boldsymbol{t}_1)), \tag{21}$$

so that the corresponding solution of the KdV hierarchy takes the form

$$u_W(t_1, \boldsymbol{t}_1) = \prod_{i=1}^{lm} \frac{2}{(t_1 - p_i(\boldsymbol{t}_1))^2}.$$

It turns out that under substitution of this expression into the KdV equation one finds [22]

$$\partial_3 p_i = 12 \sum_{j \neq i} \frac{1}{(p_i - p_j)^2}, \qquad \sum_{j \neq i} \frac{1}{(p_i - p_j)^3} = 0,$$

and this constitutes a constrained flow of the Calogero–Moser hierarchy. Similar equations are obtained by using the higher members of the KdV hierarchy. On the other hand, according to the results of the present paper, each of the functions $p_i(\boldsymbol{t}_1)$ determines a solution of the $\mathrm{hKdV}_{(3)}$ hierarchy associated with the patch

$$\boldsymbol{t}^{(i)}(\boldsymbol{t}_1) = (p_i(\boldsymbol{t}_1), \boldsymbol{t}_1).$$

Thus, from (3) the corresponding wave function is

$$\hat{\psi}_W^{(i)}(k, \boldsymbol{t}_1) = -\frac{\tau_W(\boldsymbol{t}^{(i)}(\boldsymbol{t}_1) - \epsilon(k))}{\partial_1 \tau_W(\boldsymbol{t}^{(i)}(\boldsymbol{t}_1))}.$$

For these solutions it readily follows that the equations of the $\mathrm{hKdV}_{(3)}$ hierarchy reduce to partial differential equations for the l_m functions $p_i(\boldsymbol{t}_1)$. They describe differential constraints for the *hypersurfaces* $t_1 = p_i(\boldsymbol{t}_1)$ in $\mathbb{C}^\infty$ involving several coordinates t_{2i+1}.

For example, from (10) and (18) we can see that the third equation of (11) takes the form

$$\partial_3 \partial_5 p_i = -\partial_3((\partial_3 p_i)^2 + \tfrac{1}{3}\partial_3\partial_1(\ln \partial_1 \tau)), \quad i = 0, 1, 2.$$

Using (21) we get

$$\partial_1 \ln \tau(p_i(\boldsymbol{t}_1), \boldsymbol{t}_1) = 2\sum_{j\neq i} \frac{1}{p_i(\boldsymbol{t}_1) - p_j(\boldsymbol{t}_1)},$$

and hence

$$\partial_3 \left[\partial_5 p_i + (\partial_3 p_i)^2 + \partial_3 \sum_{j\neq i} \frac{1}{p_i - p_j} \right] = 0.$$

In what concerns the higher hKdV hierarchies, they arise when manifolds of multiple zeros are present in the factorization of τ_W, so that they describe differential constraints for the *collisions* of manifolds of simple zeros.

It is interesting to notice that the solutions of the hKdV hierarchies determined in this section involve in general implicit functions $b_i(\boldsymbol{t}_m)$. This type of solutions appears also in the theory of the Harry Dym equation [23–25] which in turn is also described in the context of integrable hierarchies associated with generalized Schrödinger problems.

5. Loop groups and Birkhoff factorization

In this section we will provide a Birkhoff factorization problem for the hKdV hierarchies on the basis of the loop group description of the Grassmannian [18]. For the particular case we are interested in, it is enough to consider the Lie group GL_2 of invertible operators in $\mathbb{C}^2$ and the corresponding loop group $L\mathrm{GL}_2$ of smooth maps $g : S^1 \subset \mathbb{C} \to \mathrm{GL}_2$ with zero-index; i.e.,

$$\mathrm{Ind}\, g = \frac{1}{2\pi \mathrm{i}} \mathrm{Tr} \oint_{S_1} \mathrm{d}k \frac{\mathrm{d}g}{\mathrm{d}k} \cdot g^{-1} = 0.$$

This loop group is an infinite-dimensional Lie group – a Hilbert manifold – with Lie algebra, $L\mathfrak{gl}_2$, given by the set of loops of linear operators in $\mathbb{C}^2$. If one considers S^1 embedded in the Riemann sphere $\overline{\mathbb{C}}$ one can define the following relevant Lie subgroups of $L\mathrm{GL}_2$:

- $L^+\mathrm{GL}_2$ is the set of boundary values in S^1 of holomorphic maps from the disk $D_1(0) := \{k \in \mathbb{C} : |k| \le 1\}$ to GL_2.
- $L^-\mathrm{GL}_2$ is the set of boundary values in S^1 of holomorphic maps $\overline{\mathbb{C}}\backslash D_1(0)$ to GL_2.
- $L_1^-\mathrm{GL}_2$ is the set of loops in $L^-\mathrm{GL}_2$ normalized by unity at ∞.

Given $k \in S^1$ and $\boldsymbol{m} = (m_1, m_2) \in \mathbb{Z}^2$ let us denote

$$k^{\boldsymbol{m}} := \mathrm{diag}(k^{m_1}, k^{m_2}).$$

The Birkhoff factorization theorem states that for any loop $g \in L\mathrm{GL}_2$ there exists a unique $\boldsymbol{m}$, up to permutation, such that

$$g = g_- \cdot k^{\boldsymbol{m}} \cdot g_+$$

where $g_\pm \in L^\pm \mathrm{GL}_2$. The loops for which $\boldsymbol{m} = 0$ form a dense open subset of the identity component of $L\mathrm{GL}_2$ called the big cell of the loop group.

Let $H^{(2)} := L^2(S^1, \mathbb{C}^2)$ be the Hilbert space of square- integrable functions from S^1 into $\mathbb{C}^2$. There is a canonical isomorphism, also called lexicographic isomorphism, between $H = L^2(S^1)$ and $H^{(2)}$

$$
\begin{aligned}
&H \longleftrightarrow H^{(2)} \\
&w \longleftrightarrow \boldsymbol{w} := \begin{pmatrix} w_+ \\ w_- \end{pmatrix}
\end{aligned}
$$

given by

$$
\begin{aligned}
&w(k) = w_+(k^2) + k w_-(k^2), \\
&w_+(k^2) = \frac{w(k) + w(-k)}{2}, \qquad w_-(k^2) = \frac{w(k) - w(-k)}{2k}.
\end{aligned}
$$

This isomorphism extends to the corresponding Grassmannians, so that in what follows the symbol Gr will stand for either $\mathrm{Gr}(H)$ or $\mathrm{Gr}(H^{(2)})$ and, to avoid confusion, given a subspace W in H we will denote by $W^{(2)}$ the corresponding subspace in $H^{(2)}$. Thus, there is a natural action of $L\mathrm{GL}_2$ on Gr and, in particular, the orbit of $H_+^{(2)}$ under the action of $L\mathrm{GL}_2$ coincides with the set of subspaces $W^{(2)}$ satisfying $kW^{(2)} \subset W^{(2)}$. Observe that $W^{(2)}$ has zero virtual dimension because the loops have zero-index. As a consequence

$$
\mathrm{Gr}^{(2)} \cong L\mathrm{GL}_2 \cdot H_+^{(2)}.
$$

One can show [18] that

$$
\mathrm{Gr}^{(2)} \cong L\mathrm{GL}_2 / L^+\mathrm{GL}_2.
$$

Moreover, as it is stated in the following lemma, each stratum $\sum_m$ can be identified with a loop subgroup of $L^-\mathrm{GL}_2$ and, in particular, $\sum_0 \cong L_1^-\mathrm{GL}_2$. To formulate the next results some previous notation is required. For a given m we define its parity $p(m)$, its associated Pauli matrix $\sigma_{p(m)}$ and the vector $\boldsymbol{m}$ as follows:

$$
\begin{aligned}
&p(m) := \frac{1 - (-1)^m}{2} \in \mathbb{Z}_2, \\
&\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&\boldsymbol{m} := (-1)^m \left[\frac{m+1}{2}\right](-1, 1),
\end{aligned}
$$

where $[a]$ is the entire part of a. The following Lie subgroups are also required:

- $N^-\mathrm{GL}_2$ is the set of loops in $L^-\mathrm{GL}_2$ such that at $k = \infty$ differ from the identity in a strictly upper triangular matrix.
- $N^+\mathrm{GL}_2$ is the set of loops in $L^+\mathrm{GL}_2$ such that at $k = 0$ are lower triangular.
- $L_{\boldsymbol{m}}^-\mathrm{GL}_2 := k^{\boldsymbol{m}} L_1^-\mathrm{GL}_2 k^{-\boldsymbol{m}} \cap N^-\mathrm{GL}_2$.
- $L_{\boldsymbol{m}}^+\mathrm{GL}_2 := k^{\boldsymbol{m}} L_1^-\mathrm{GL}_2 k^{-\boldsymbol{m}} \cap N^+\mathrm{GL}_2$.

Notice that for $\boldsymbol{m} = \mathbf{0}$ one gets

$$L^-_{\boldsymbol{m}=\mathbf{0}}\mathrm{GL}_2 = L^-_1\mathrm{GL}_2, \qquad L^+_{\boldsymbol{m}=\mathbf{0}}\mathrm{GL}_2 = \{1\}.$$

The following result is already proven in [18]. Nevertheless, we are including a proof as it contains some essential ingredients for our remaining discussion.

Lemma 1.

$$\Sigma_m \cong L^-_m\mathrm{GL}_2.$$

Proof. Recall that the stratum $\sum_m$ has a distinguished subspace

$$H_m = \overline{\mathbb{C}\{k^{2n}k^{-m}, k^{2n}k^{m+1}\}}_{n\geq 0},$$

such that for each W in $\sum_m$ the orthogonal projection $W \to H_m$ is an isomorphism. Therefore there exists a unique basis, the canonical basis [18], $\{k^{2n}w_0, k^{2n}w_1\}_{n\geq 0}$ of W satisfying

$$w_0(k) = k^{-m} + v_0(k), \qquad w_1(k) = k^{m+1} + v_1(k),$$

with $v_0, v_1 \in H_m^{\perp}$, $v_0 = \mathrm{O}(k^{-m-1})$ and $v_1 = \mathrm{O}(k^{m-1})$.

The subspace $H_m^{(2)} \subset H^{(2)}$ corresponding to H_m under the lexicographic isomorphism is

$$H_m^{(2)} = k^{\boldsymbol{m}} \cdot H_+^{(2)},$$

while the subspace $W^{(2)}$ corresponding to W is

$$W^{(2)} = \tilde{g}_W \cdot H_+^{(2)}.$$

Here $\tilde{g}_W \in L\mathrm{GL}_2$ is a loop constructed in terms of generators w_0 and w_1 as follows:

$$\tilde{g}_W = (\boldsymbol{w}_0, \boldsymbol{w}_1) \cdot \sigma_{p(m)}.$$

Observe that

$$\tilde{g}_W = g_W \cdot k^{\boldsymbol{m}} \quad \text{and} \quad g_W \in L^-_{\boldsymbol{m}}\mathrm{GL}_2,$$

so that we have

$$W^{(2)} = g_W \cdot H_m^{(2)}.$$

Hence each subspace in Σ_m determines an element of $L^-_m\mathrm{GL}_2$. Reciprocally, any loop in $L^-_m\mathrm{GL}_2$ give rise to a subspace in Σ_m. Therefore,

$$\Sigma_m = L^-_m\mathrm{GL}_2 \cdot H_m^{(2)}. \qquad \Box$$

We are now ready to relate the hKdV hierarchies with Birkhoff factorizations. To this end we notice that under the lexicographic isomorphism the multiplication operator by ψ_0 transforms into the multiplication operator by the matrix function

$$\psi_0(k, t) := \exp\left(\sum_{n\geq 0} t_{2n+1} k^n J(k)\right),$$

where

$$J(k) := \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix}.$$

Theorem 4. *Given $g_0 \in L\mathrm{GL}_2$, let W be the subspace in H corresponding to $g_0 \cdot H_+^{(2)}$ and let τ_W be its associated τ-function. Then, for every patch* (14) $\boldsymbol{t}_m \mapsto \boldsymbol{t}$ *of the zero set of τ_W, the Birkhoff factorization*

$$\psi_0^{-1} \cdot g_0 = \Phi_-^{-1} \cdot k^{\boldsymbol{m}} \cdot \Phi_+, \tag{22}$$

where

$$\psi_0(k, \boldsymbol{t}(\boldsymbol{t}_m)) := \exp\left(\left(b(k, \boldsymbol{t}_m) + \sum_{n\geq m} t_{2n+1} k^n\right) J(k)\right),$$

$$\Phi_-(\boldsymbol{t}_m) \in L_{\boldsymbol{m}}^- \mathrm{GL}_2, \qquad \Phi_+(\boldsymbol{t}_m) \in L^+ \mathrm{GL}_2$$

describes the flows of the hKdV_{2m+1} *hierarchy on the Grassmannian.*

Proof. By defining

$$\Phi_-^{-1}(\boldsymbol{t}_m) := g_{W(\boldsymbol{t}(\boldsymbol{t}_m))} \in L_m^- \mathrm{GL}_2,$$

and by using Lemma 1 we can write

$$W(\boldsymbol{t}(\boldsymbol{t}_m)) = \Phi_-^{-1}(\boldsymbol{t}_m) \cdot H_m^{(2)}.$$

Furthermore, $W = g_0 \cdot H_+^{(2)}$ for some loop g_0 so that

$$W(\boldsymbol{t}(\boldsymbol{t}_m)) = \psi_0^{-1}(\boldsymbol{t}(\boldsymbol{t}_m)) \cdot g_0 \cdot H_+^{(2)},$$

and we arrive to the general Birkhoff factorization problem (22). □

Now, we analyze the infinitesimal aspects of the Birkhoff factorization problem (22) that will lead us to Sato's type equations and the zero- curvature representation of the hKdV_{2m+1} hierarchy.

We first consider the splitting

$$L\mathfrak{gl}_2 = \mathrm{Ad}k^{\boldsymbol{m}}(L^+\mathfrak{gl}_2) \oplus \mathrm{Ad}k^{\boldsymbol{m}}(L_1^-\mathfrak{gl}_2)$$

whose resolution of the identity is

$$\mathrm{id} = P_+^{\boldsymbol{m}} + P_-^{\boldsymbol{m}}.$$

Observe that this resolution is obtained from the standard one by conjugation, $P_{\pm}^{m} = \mathrm{Ad}k^{m} \circ P_{\pm} \circ \mathrm{Ad}k^{-m}$. We further split the Lie subalgebra $\mathrm{Ad}k^{m}(L_{1}^{-}\mathfrak{gl}_2)$ as

$$\mathrm{Ad}k^{m}(L_{1}^{-}\mathfrak{gl}_2) = L_{m}^{+}\mathfrak{gl}_2 \oplus L_{m}^{-}\mathfrak{gl}_2,$$

and the corresponding resolution is

$$P_{-}^{m} = \pi_{+}^{m} + \pi_{-}^{m}.$$

Here $\mathrm{Ad}\psi$ denotes the adjoint action of ψ in the Lie algebra $\mathrm{Ad}\psi(X) = \psi \cdot X \cdot \psi^{-1}$. The set $L_{m}^{+}\mathfrak{gl}_2$ can be identified with the Schubert cell C_m in $\mathrm{Gr}^{(2)}$ which complements the stratum Σ_m in the Grassmannian [18]. Thus, in the previous splitting one can associate π_{+}^{m} with the Schubert cell C_m and π_{-}^{m} with the stratum Σ_m.

Theorem 5. *The infinitesimal version of the Birkhoff factorization* (22) *decouple into the following Sato's equations for Φ_-:*

$$\pi_{+}^{m}((\partial_{2n+1}b + k^n)\mathrm{Ad}\Phi_-(J)) = 0, \quad \textit{Schubert cell},$$
$$\pi_{-}^{m}((\partial_{2n+1}b + k^n)\mathrm{Ad}\Phi_-(J)) = \partial_{2n+1}\Phi_- \cdot \Phi_-^{-1}, \quad \textit{stratum},$$

and an equation linking the Lax operators

$$L_{2n+1} := -\partial_{2n+1}\Phi_+ \cdot \Phi_+^{-1}, \quad n \geq m$$

with Φ_-:

$$L_{2n+1} = P_+((\partial_{2n+1}b + k^n)\mathrm{Ad}k^{-m}\Phi_-(J)).$$

where $n \geq m$.

Proof. By taking right derivatives on (22) we arrive to

$$\begin{aligned}&\partial_{2n+1}\Phi_- \cdot \Phi_-^{-1} - (k^n + \partial_{2n+1}b)\mathrm{Ad}\Phi_-(J)\\ &\quad = \mathrm{Ad}k^{m}(\partial_{2n+1}\Phi_+ \cdot \Phi_+^{-1}).\end{aligned} \tag{23}$$

One can readily notice that a Gel'fand–Dikii argument [26] may be applied in this equation. This is so since $\partial_{2n+1}\Phi_- \cdot \Phi_-^{-1}$ belongs to the Lie algebra $\mathrm{Ad}k^{m}(L_{1}^{-}\mathfrak{gl}_2)$, while the RHS of (23) belongs to the Lie algebra $\mathrm{Ad}k^{m}(L^{+}\mathfrak{gl}_2)$:

$$P_{+}^{m}((\partial_{2n+1}b + k^n)\mathrm{Ad}\Phi_-(J)) = -\mathrm{Ad}k^{m}(\partial_{2n+1}\Phi_+ \cdot \Phi_+^{-1}), \tag{24}$$
$$P_{-}^{m}((\partial_{2n+1}b + k^n)\mathrm{Ad}\Phi_-(J)) = \partial_{2n+1}\Phi_- \cdot \Phi_-^{-1}, \tag{25}$$

the first equation gives the zero-curvature representation while the second is the Sato equation for Φ_-.

We notice that Eq. (25) splits into two parts

$$\pi_{+}^{m}((\partial_{2n+1}b + k^n)\mathrm{Ad}\Phi_-(J)) = 0,$$
$$\pi_{+}^{m}((\partial_{2n+1}b + k^n)\mathrm{Ad}\Phi_-(J)) = \partial_{2n+1}\Phi_- \cdot \Phi_-^{-1}. \qquad \square$$

For the Lax operators L_{2n+1}, $n \geq m$, one gets zero-curvature equations

$$\partial_{2r+1} L_{2s+1} - \partial_{2s+1} L_{2r+1} + [L_{2r+1}, L_{2s+1}] = 0, \quad r, s \geq m.$$

Moreover, because of Eq. (24) one deduces that L_{2r+1} belongs to $L^+\mathfrak{gl}_2$ and also that it is a polynomial in k.

We analyse now the Lax operator L_{2m+1}. To this end we first seek for the generators $(w_0(\boldsymbol{t}_m), w_1(\boldsymbol{t}_m))$ of a canonical basis of $W(\boldsymbol{t}(\boldsymbol{t}_m))$. It is easy to see that the function w_0 can be identified with the function $\hat{\psi}_W$ given in (15). We also know that there exists a unique function $\hat{\phi}_W(\boldsymbol{t}(\boldsymbol{t}_m))$ that under the orthogonal projection $W \to H_m$ transforms as $\hat{\phi}_W \mapsto k^{m+1}$. This function is just the generator w_1. One can show that $\phi_W := \psi_0(\boldsymbol{t}(\boldsymbol{t}_m))\hat{\phi}_W$ can be written as

$$\phi_W = (\partial_x - p(\lambda, \boldsymbol{t}_m))\psi_W,$$

where $x := t_{2m+1}$ and

$$p := \sum_{j=0}^{m} \lambda^j p_j(\boldsymbol{t}_m)$$

is a polynomial in λ with coefficients p_j which are differential polynomials in the coefficients a's and b's.

Consider next the matrix wave-function

$$\Psi := \psi_0 \cdot \Phi_-^{-1} \cdot k^m = g_0 \cdot \Phi_+^{-1} = (\psi_W, \phi_W) \cdot \sigma_{p(m)}$$

the columns of which being the vectors ψ_W and $\phi_W = (\partial_x - p)\psi_W$ in the proper order, where we are using the notation $x := t_{2m+1}$. In computing $\partial_x \Psi$ we need

$$\partial_x \phi_W = -p\partial_x \psi_W + (u - \partial_x p)\psi_W$$

which derives from the energy-dependent Schrödinger equation (2). Then, one can check that

$$\Psi^{-1} \cdot \partial_x \Psi = L_{2m+1} = \sigma_{p(m)} \cdot \begin{pmatrix} p & u - \partial_x p - p^2 \\ 1 & -p \end{pmatrix} \cdot \sigma_{p(m)}.$$

Moreover, a natural gauge transformation is provided by

$$\tilde{\Psi} := \Psi \cdot \Omega = (\psi_W, \partial_x \psi_W) \cdot \sigma_{p(m)},$$

with

$$\Omega := \sigma_{p(m)} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \cdot \sigma_{p(m)}.$$

The gauge transformed Lax operator is

$$\hat{L}_{2m+1} = \sigma_{p(m)} \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix} \cdot \sigma_{p(m)}.$$

For the t_{2n+1}-evolution we proceed in a similar way by using now Eqs. (3) and (7). Thus, we finally obtain the following expressions for the Lax operators L_{2n+1}, $n > m$

$$L_{2n+1} = \sigma_{p(m)} \cdot \begin{pmatrix} \beta_n p - \frac{1}{2}\partial_x \beta_n & -\frac{1}{2}\partial_x^2 \beta_n + (\partial_x \beta_n) p + \beta_n (u - p^2) - \partial_{2n+1} p \\ \beta_n & -(\beta_n p - \frac{1}{2}\partial_x \beta_n) \end{pmatrix} \cdot \sigma_{p(m)},$$

while for the gauge transformed operators we get

$$\hat{L}_{2n+1} = \sigma_{p(m)} \cdot \begin{pmatrix} -\frac{1}{2}\partial_x \beta_n & -\frac{1}{2}\partial_x^2 \beta_n + \beta_n u \\ \beta_n & \frac{1}{2}\partial_x \beta_n \end{pmatrix} \cdot \sigma_{p(m)}.$$

Notice that the operators

$$\mathcal{L}_{2n+1} = \begin{pmatrix} -\frac{1}{2}\partial_x \beta_n & -\frac{1}{2}\partial_x^2 \beta_n + \beta_n u \\ \beta_n & \frac{1}{2}\partial_x \beta_n \end{pmatrix},$$

do satisfy the zero-curvature equations if and only if the corresponding hKdV hierarchy holds. This representation was first introduced for the KdV hierarchy by Novikov [27].

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